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On Spinning Particles

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There exist classical systems whose canonical quantization yields relativistic wave equations. As a constructive proof, the classical mechanics of a translating-rotating five-frame is considered. Its quantization yields the Dirac, Weyl, Klein-Gordon, Maxwell-Proca, and higher spin equations, together with a rotational mass spectrum for the states predicted.

KEY WORDS: canonical quantization; spinning particles; relativistic wave equation

1. INTRODUCTION

In physical theory, one seeks to explain the complexity and diversity of things in terms of the lawful organization of a few kinds of units. These units are endowed with the least structure compatible with their perceived existence. Thus it was natural that the electron was regarded for some time after its discovery as a massive charged point in a three-dimensional manifold. The first indication of greater structure came from spectroscopic evidence, and led Uhlenbeck and Goudsmit (1926) to suggest that a spin is associated with the former geometrical point. Shortly thereafter, Pauli (1927) formulated a quantum theory of spinning particles, utilizing the spin $1/2$ representation of the three-dimensional rotation group to account for the observed quantization of the spin.

The chief deficiency of Pauli's theory was its nonrelativistic nature, and this was remedied in the epochal discovery of Dirac (1928). His equation, which may be written

$$
\begin{gathered}
(\gamma^{\mu}\,\partial_{\mu} + m)\psi = 0 \\
[\gamma^{\mu}, \gamma^{\nu}]_{+} = -2g^{\mu\nu}
\end{gathered}
\tag{1}
$$

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found immediate confirmation in the fine structure of hydrogen, and later in the successes of quantum electrodynamics. Its position today is an ambivalent one. While exhaustively understood from many mathematical and physical points of view (Corson, 1953; Rose, 1961; Schweber, 1961), it is not well understood kinematically. The reason is that Eq. (1) stands as a system of partial differential equations, the particle nature being only inferred, via the construction of tensors commonly associated at the quantum level with particles. In addition, there are grave conceptual difficulties, like zitterbewegung, unbounded negative energy spectrum, etc., connected with the anomalous behavior of the inferred particle.

The situation has led many people (Grossman and Peres, 1963) to construct classical models bearing a more or less close relationship to Dirac's equation, but these attempts have met only partial success. Moreover, classical systems lead almost invariably to second order equations, while Dirac's is a first order equation with an essential dependence on anti-commutation relations, a concept with no ready classical analogue. These facts have led most to believe that the electron is an intrinsically quantum mechanical system, admitting no classical mechanical description.

It has been suggested (Finkelstein, 1955) that Dirac's equation does indeed result from the quantization of some underlying classical system, but that it represents only a direct summand in the ensuing scheme, the remaining summands describing states of higher and lower spin, flowing equally naturally from the underlying theory and obeying other irreducible equations. Nevertheless, Dirac's equation may be expected to bear a closer resemblance to the underlying theory than these others. This is by analogy with the theory of group representations, where it is known that the spinor representations are the fundamental ones.

"Classical system" and "quantization" are meant in the most literal and conservative sense; namely, that exemplified by the first system to suffer modern quantization, Schrödinger's hydrogen atom. One must start with a well-defined configuration space and impose on it a Lagrangian. From this is derived the infinitesimal generator of the motion. One passes next to the quantization of the system by making the usual operator replacements, leading to a dynamical equation and the associated apparatus of the quantum theory. One may wish, in addition, that the theory be invariant under some group. For this it suffices that the Lagrangian be invariant; in the Hamiltonian formulation, it is necessary that the Lie algebra of the invariance group, realized in terms of canonical variables, be included among the constants of the motion.

In attempting to find such a system, two features of Dirac's theory have seemed salient. First, it is clear that in some sense one is dealing with a spinning object whose translational motion is coupled to its spin. Second, it has long been known [(Klein, 1936; Bhabha, 1945) More generally in n dimensions, the n Clifford numbers, and their (n2) commutators, span the Lie algebra of a pseudo-orthogonal group in $n + 1$ dimensions.] that the four Dirac matrices $\gamma'_\mu \equiv \frac{1}{2}\, \gamma_\mu$

generate under repeated commutation of the Lie algebra $o(3, 2; R)$ of the group $O(3, 2; R)$:

$$[\gamma'_\mu, \gamma'_\nu] \equiv \gamma'_{\mu\nu} \tag{2a}$$

$$[\gamma'_{\mu\nu}, \gamma'_\sigma] = g_{\mu\sigma}\,\gamma'_\nu - g_{\nu\sigma}\,\gamma'_\mu \tag{2b}$$

$$[\gamma'_{\mu\nu}, \gamma'_{\sigma\tau}] = g_{\mu\sigma}\,\gamma'_{\nu\tau} + g_{\nu\tau}\,\gamma'_{\mu\sigma} - g_{\mu\tau}\,\gamma'_{\nu\sigma} - g_{\nu\sigma}\,\gamma'_{\mu\tau} \tag{2c}$$

Equation (2c) states that the $\gamma'_{\mu\nu}$ generate the Lie algebra, hlg, of the homogeneous Lorentz group, HLG. Equations (2b) and (2c) state that hlg is reductive in $o(3, 2; R)$.

Defining $\gamma'_{5\mu} \equiv \gamma'_u \equiv -\gamma'_{\mu 5}$, Eqs. (2b) and (2c) may be combined into the single equation

$$[\gamma'_{ab}, \gamma'_{cd}] = g_{ac}\,\gamma'_{bd} + g_{bd}\,\gamma'_{ac} - g_{ad}\,\gamma'_{bc} - g_{bc}\,\gamma'_{ad}. \tag{3}$$

Here $\mu, \nu = 0, 1, 2, 3$; $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$; $a, b = 5, 0, 1, 2, 3$; $g_{ab} = \mathrm{diag}\,(1,\ 1,\ -1,\ -1,\ -1)$. $0(3, 2; R)$, also known as the $3 + 2$ de Sitter group, is the 10-parameter group of real 5×5 matrices leaving invariant the form $g_{ab}\,x^a\,x^b$. In Cartan's complex classification this group is related to the locally isomorphic groups B_2 and C_2.

2. CLASSICAL THEORY

2.1. Configuration Space and Covariance

The configuration space is the principle fiber bundle (Chern, 1966; Nomizu, 1956; Steenrod, 1951) whose base is Minkowski space-time and whose fiber is $0(3, 2; R)$ conceived of as a frame manifold. The dynamical invariance group is the inhomogeneous Lorentz group, ILG.

The physical meaning may be clarified by reference to the ordinary top, whose internal configuration space is the manifold of $0(3; R)$. Points of this manifold correspond to orthonormal frames, representing pure orientations. Similarly, a point of $0(3, 2; R)$ in general corresponds to a distended frame, so the fiber may be thought of as the manifold of possible orientation-deformations contemplated by the theory.[3]

We wish to do classical mechanics in this space; i.e., consider maps of the real line into the bundle, generated by Euler-Lagrange or Hamiltonian equations. One wishes these maps to assemble themselves into equivalence classes, within each of which a realization of the dynamical invariance group acts transitively. This is the requirement of covariance. As stated, the present theory is built on ILG.

[3] The notion of manifolds of kinematic or geometric objects is chiefly due to Plücker, who, embittered by neglect, left geometry for twenty years to become a founder of experimental spectroscopy.

Covariance may be secured by establishing an action of ILG on the bundle and later choosing an invariant Lagrangian. It is assumed that ILG acts on space-time in the usual manner; e.g., X^μ is a 4-vector transforming under ILG, while $\dot{x}^\mu \equiv \frac{dx^\mu}{d\sigma}$ (σ being an arbitrary invariant path parameter) transforms under the homogeneous Lorentz group, HLG. We must impress an action of ILG on the fiber. We postulate that translations have no effect, so it remains to realize HLG on the fiber. To this end, consider the commutation relations. Equations (2b) and (2c) state, as noted above, that HLG is a reductive subgroup of $0(3, 2; R)$. These equations may now be read in the following way. Let $A \in hlg,\ B \in o(3, 2; R)$. Then $[A, B]$ gives the change in the tangent vector B induced by the infinitesimal inner automorphism generated by A. Thus Eq. (2b) says that four of the generators of $o(3, 2; R)$ transform among themselves as a 4-vector, the remaining generators, by Eq. (2c) transforming among themselves as a bivector. Hence HLG is realized as an adjoint group of $o(3, 2; R)$, and as a group of inner automorphisms of $O(3, 2; R)$.

2.2. The Spinning Object

Consider next a rotating 5-frame. With respect to a fiducial system of five axes (a tetrad of vectors tangent to the space-time coordinates, a fifth in the normal direction), the frame is specified by a 5×5 matrix f. (Changing the fiducial axes changes f by inner automorphism.) The matrix elements of f are functions of ten Euler angles q^{ab} on the manifold $O(3, 2; R)$. The motion of f is determined by a path of this manifold; i.e., $q^{ab} = q^{ab}(\sigma)$, σ an arbitrary invariant path parameter.

As in the theory of the ordinary top, it is natural to consider the Darboux-Cartan matrix $\Omega(q, \dot{q}) = \dot{f} f^{-1}$, which lies in the dual to the Lie algebra. Denote the matrix elements of Ω by $\Omega^a_b(q, \dot{q}) = w^a_{bcd}(q)\, \dot{q}^{cd}$. The functions $w^a_{bcd}(q)$ are associated with the 1-forms $w^{ab}_{cd}(q)\, dq^{cd}$, and these with their duals, the vector fields of left translation $L_{ab} = \overline{w}^{cd}_{ab}(q)\, r_{cd}$. The symbol $\overline{w}$ denotes the inverse-transpose of the matrix w, and $r_{cd} \equiv \frac{\partial}{\partial q^{cd}}$. These vector fields constitute the Lie algebra of $O(3, 2; R)$ acting on its own manifold as a group of motions, hence $[L_{ab}, L_{cd}] = C^{ef}_{\text{abcd}}\, L_{ef}$, where C is the structure tensor of $o(3, 2; R)$. But this must be equivalent to Eq. (3). Hence we may regard the Dirac matrices γ'_{ab} as 4-dimensional representations of the infinite-dimensional operators L_{ab}.

2.3. The Action

For the action we take

$$S = \int d\sigma\, m_0 c\big(\dot{x}^\mu\, \dot{x}_\mu + \alpha R\, \dot{x}^\mu\, w_{5\mu\, ab}\, \dot{q}^{ab} + \beta R^2 w_{\mu\nu ab}\, w^{\mu\nu}_{cd}\, \dot{q}^{ab}\, \dot{q}^{cd}\big)^{1/2}. \quad (4)$$

The purpose of the square root is to render the Lagrangian homogeneous of degree one, so that the equations of motion are sensitive to reparametrization. R is a quantity of the dimension of a length, which plays the role of a radius of gyration. The symbols α and β are dimensionless coupling constants. The first term in the Lagrangian pertains to translational motion in space-time, the last term to rotational motion in the fiber, and the middle term to translation-rotation coupling. For simplicity we have assumed an inertia tensor that is isotropic in four dimensions and independent of the angular velocities. If R is set equal to zero we recover the Lagrangian for a structureless, free relativistic particle of mass m_0. It may be anticipated that R may be written as a dimensionless constant times the Compton wavelength $\hbar/m_0 c$, so that $R \to 0$ is achieved by $\hbar \to 0$. In this sense, the spin of this classical particle owes its existence to the finite value of Planck's constant.

2.4. Hamiltonian Formulation

While the Euler-Lagrange equations derived from the action are sufficient for many purposes, it is useful in the classical theory, and necessary for conventional quantum theory, to display the motion as the evolution of a contact transformation. This is accomplished by the Hamiltonian formalism.

The canonical momenta are

$$p_\mu \equiv \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = \frac{(m_0 c)^2}{2\mathcal{L}}(2\dot{x}^\mu + \alpha\, R w_{5\mu\, ab}\, \dot{q}^{ab}) \tag{5a}$$

$$r_{cd} \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}^{cd}} = \frac{(m_0 c)^2}{2\mathcal{L}} \left(\alpha\, R\, \dot{x}^\mu\, w_{5\mu\, cd} + 2\beta\, R^2\, w_{\mu\nu ab}\, w^{\mu\nu}_{cd}\, \dot{q}^{ab}\right). \tag{5b}$$

Multiplying this last equation by $\overline{w}$ yields

$$L_{5\mu} \equiv \overline{w}^{ab}_{5\mu}\, r_{ab} = \frac{(m_0 c)^2}{2\mathcal{L}}\, \alpha\, R\, \dot{x}^\mu \tag{6a}$$

$$L_{\mu\nu} \equiv \overline{w}^{ab}_{\mu\nu}\, r_{ab} = \frac{(m_0 c)^2}{\mathcal{L}}\, \beta\, R^2\, w_{\mu\nu ab}\, \dot{q}^{ab}. \tag{6b}$$

The canonical momenta are not all independent since we have the constraint

$$L^\mu\, p_\mu - \frac{L^\mu\, L_\mu}{\alpha\, R} + \frac{\alpha}{4\beta\, R}\, L^{\mu\nu}\, L_{\mu\nu} = \frac{\alpha\, R}{4}\, (m_0 c)^2$$

where we have written, for simplicity, L^μ for $L^{5\mu}$. If we take $\beta = \frac{-\alpha^2}{4}$, the constraint assumes the simple form

$$I \equiv L^\mu\, p_\mu - \frac{L_{ab}\, L^{ab}}{\alpha R} - \frac{\alpha R}{4}\, (m_0 c)^2 = 0$$

involving the Casimir operator $L_{ab}\,L^{ab}$. It is now no loss of generality to take $\alpha = 2$. With these specializations,

$$I = L^{\mu}\,p_{\mu} - \frac{L_{ab}\,L^{ab}}{2R} - \frac{R}{2}\,(m_0 c)^2 \tag{7}$$

and the Lagrangian of the theory is

$$\mathcal{L} = m_0 c\,\left(\dot{x}^{\mu}\,\dot{x}_{\mu} + 2R\,\dot{x}^{\mu}\,w_{5\mu ab}\,\dot{q}^{ab} - R^2\,w_{\mu\nu ab}\,w^{\mu\nu}_{cd}\,\dot{q}^{ab}\,\dot{q}^{cd}\right)^{1/2} \tag{8}$$

in which the subtractive contribution of the rotational form is nominal, since this form is indefinite.

By Hamilton's principle the equations of motion are obtained by extremalizing the action

$$S = \int d\sigma(p_{\mu}\,\dot{x}^{\mu} + r_{ab}\,\dot{q}^{ab} - H - \lambda\,I)$$

where now free variations of both the coordinates and momenta are contemplated, subject only to the limitation implied by the constraint. This limitation is achieved by picking up the constraint with a Lagrange multiplier λ. Since the Lagrangian was homogeneous of degree one, the Hamiltonian H vanishes identically, with the consequence that the motion is generated solely by the constraint I [The procedure being followed is clearly explained in Lanczos (1949)].

The variation then yields the equations of motion

$$\begin{aligned} \dot{x}^{\mu} &= \lambda\,\frac{\partial I}{\partial p_{\mu}} \qquad & \dot{p}^{\mu} &= -\lambda\,\frac{\partial I}{\partial x^{\mu}} \\ \dot{q}^{ab} &= \lambda\,\frac{\partial I}{\partial r_{ab}} \qquad & \dot{r}_{ab} &= -\lambda\,\frac{\partial I}{\partial q^{ab}} \\ & \text{and } I = 0. \end{aligned} \tag{9}$$

We may now choose a path parameter σ' so that $\lambda(\sigma') = 1/m_0 c$ identically, and the dot in these equations now denotes differentiation with respect to σ'. By Eq. (6a), the Lagrangian is identically equal to $Rm_0 c$, having the dimension of action.

Defining $I' \equiv I/m_0 c$, we may introduce Poisson brackets in the usual manner, so that the σ' rate of change of any dynamical quantity Y may be written as $\dot{Y} = -[I', Y]_{P.B.}$.

The description of the motion may be referred to quantities z other than σ' by means of the formula

$$\frac{dY}{dz} = \frac{dY/d\sigma'}{dz/d\sigma'} = \frac{[I',\ Y]_{P.B.}}{[I', z]_{P.B.}}. \tag{10}$$

We may do this for the coordinates x^μ in particular, with the result that the generators of space-time translation are given by

$$\frac{d}{dx^\mu} = \frac{[I',\ ]_{P.B.}}{[I', x^\mu]_{P.B.}} = -L_\mu^{+1}\,[I,\ ]_{P.B.}$$
$$= -\left[L_\mu^{-1}\, I,\ \right]_{P.B.} \equiv ad\left(-L_\mu^{-1},\ I\right), \tag{11}$$

since $I = 0$. On quantization, these pass over to the Heisenberg equations of motion.

2.5. Motion of a Free Particle

We study the projection of the motion onto the base space. Evaluating Poisson brackets,

$$\dot{p}_\mu = 0, \qquad \text{so} \quad p_\mu = \text{const.} \tag{12a}$$

$$\dot{x}^\mu = \frac{L^\mu}{m_0 c} \tag{12b}$$

$$\dot{L}^\mu = -\frac{p_\alpha\, L^{\alpha\mu}}{m_0 c} \tag{12c}$$

$$\dot{L}^{\alpha\,\mu} = \frac{p^{[\alpha}\ L^{\mu]}}{m_0 c} \tag{12d}$$

Combining the last two equations yields

$$\ddot{L}^\mu + \overline{p}^2\, \Theta^\mu_\alpha\, L^\alpha = 0$$
$$\overline{p}^\mu \equiv \frac{p_\mu}{m_0 c}, \quad \Theta^\mu_\alpha \equiv \left(\partial^\mu_\alpha - \frac{\overline{p}^\mu\, \overline{p}_\alpha}{\overline{p}^2}\right) \tag{13}$$

The quantity Θ^μ_α is a projection operator, which acts on a vector by annihilating its component along p_μ, leaving untouched its transverse part. Equation (13) then states that the transverse part of L^μ whirls around p_μ at a frequency $\overline{p} \equiv (\overline{p}_\alpha\, \overline{p}^\alpha)^{1/2}$. Integrating,

$$L^\mu(\sigma') = \overline{p}^\mu \left(\frac{\overline{p}_\alpha\, L^\alpha(0)}{\overline{p}^2}\right) + \cos\, \overline{p}\,\sigma'\ \Theta^\mu_\alpha\ L^\alpha(0) + \sin\, \overline{p}\,\sigma'\, \frac{\Theta^\mu_\alpha\, \dot{L}^\alpha(0)}{\overline{p}}$$

$$= \overline{p}^\mu \left(\frac{\overline{p}_\alpha\ L^\alpha(0)}{\overline{p}^2}\right) + \left[\left(\Theta^\mu_\alpha\ L^\alpha(0)\right)^2 + \left(\frac{\Theta^\mu_\alpha\ \overline{p}_\beta\ L^{\beta\,\alpha}(0)}{\overline{p}}\right)^2\right]^{1/2}$$
$$\times \cos\left(\overline{p}\,\sigma' + \varphi^\mu\right)$$

$$\varphi^\mu \equiv \tan^{-1}\left(\frac{\Theta^\mu_\alpha\ p_\beta\ L^{\beta\,\alpha}(0)}{\overline{p}\ \Theta^\mu_\alpha\ L^a(0)}\right)$$

where we have used the relation $\dot{L}^{\alpha}(0) = -\overline{p}_{\beta}\, L^{\beta\,\alpha}(0)$ from Eq. (12c), and the fact that $\overline{p}_{\alpha}\, \overline{p}_{\beta}\, L^{\alpha\,\beta}(0) = 0$, since $L^{\alpha\beta}$ is skew in its indices.

It is desirable to replace the initial values with the constants of the motion to which they are equal. One notes that $\frac{d}{d\sigma'}(m_0 c\, \frac{x^{\alpha}\, \overline{p}_{\alpha}}{L^{\alpha}\, \overline{p}_{\alpha}}) = 1$, so $\sigma' = m_0 c\, \frac{x^{\alpha}\, \overline{p}_{\alpha}}{L^{\alpha}\, \overline{p}_{\alpha}}$, up to an additive constant. Defining $\theta \equiv \overline{p}\, m_0 c\, \frac{x^{\alpha}\, \overline{p}_{\alpha}}{L^{\alpha}\, \overline{p}_{\alpha}}$, one checks that

$$B^{\mu} \equiv \Theta^{\mu}_{\alpha} \left(-\sin\theta\, L^{\alpha} + \cos\theta\, \frac{\overline{p}_{\beta}\, L^{\beta\,\alpha}}{\overline{p}} \right) \tag{14a}$$

$$C^{\mu} \equiv \Theta^{\mu}_{\alpha} \left(\cos\theta\, L^{\alpha} + \sin\theta\, \frac{\overline{p}_{\beta}\, L^{\beta\,\alpha}}{\overline{p}} \right) \tag{14b}$$

are constants of the motion. Evaluating them at $\sigma' = 0$, $B^{\mu} = \Theta^{\mu}_{\alpha}\, \frac{\overline{p}_{\beta}\, L^{\beta\,\alpha}(0)}{\overline{p}}$, and $C^{\mu} = \Theta^{\mu}_{\alpha}\, L^{\alpha}(0)$; so $\varphi^{\mu} = \tan^{-1}(B^{\mu}/C^{\mu})$. Also, $\overline{p}_{\alpha}\, L^{\alpha}$ is a constant of the motion. Hence

$$L^{\mu}(\sigma') = \overline{p}^{\mu} \left(\frac{\overline{p}_{\alpha}\, L^{\alpha}}{\overline{p}^{2}} \right) + [(B^{\mu})^{2} + (C^{\mu})^{2}]^{1/2}\, \cos(\overline{p}^{\sigma'} + \varphi^{\mu}). \tag{15}$$

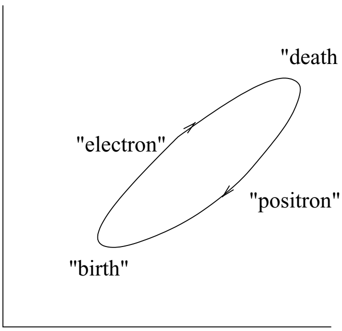
Integrating Eq. (12b),

$$x^{\mu}(\sigma') = \frac{\overline{p}^{\mu}}{m_0 c} \left(\frac{\overline{p}_{\alpha}\, L^{\alpha}}{\overline{p}^{2}} \right) \sigma' + \frac{[(B^{\mu})^{2} + (C^{\mu})^{2}]^{1/2}}{\overline{p}\, m_0 c}\, \sin(\overline{p}\, \sigma' + \varphi^{\mu}) \tag{16}$$

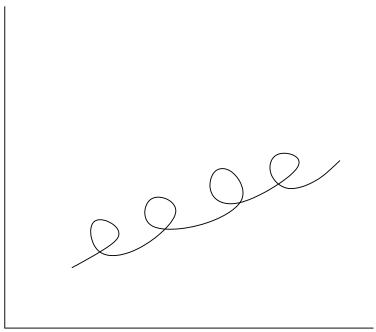
up to a constant. There are two interesting cases.

Case 1: $\boldsymbol{\overline{p}_{\alpha}\, L^{\alpha} = 0}$. Then $x^{\mu}(\sigma')$ describes an ellipse. As illustrated in Fig. 1, the two sides of the ellipse represent counter-streaming currents (Cf. (Feynman, 1949)) of charge and energy; so that, taken as a whole, it corresponds to a zero energy, neutral entity, a classical "vacuum fluctuation."

Case 2: $\boldsymbol{\overline{p}_{\alpha}\, L^{\alpha} \neq 0}$. The motion is the elliptical motion of Case 1, on which a constant drift is superimposed. The orbit is a trochoid; cf. Fig. 2. Due to the



(a) Fig. 1



(b) Fig. 2

particle's corkscrewing back into the past (or back into the future), the function $\vec{x}\,(t)$ may be multi-valued, so the position of the particle is not a well-defined concept. This is the primitive, classical reason for the general breakdown of the probability interpretation of relativistic wave equations. By continuity of the orbit, the number of different $\vec{x}\,(t)$ corresponding to given t is odd for almost all t, ensuring charge conservation. The orbit would be interpreted by an external observer as that of a particle of unit charge, with a multipole structure.

The notion of a particle corkscrewing back into the past, as it moves generally into the future, may at first sight appear to contradict Lorentz invariance. This seeming contradiction is dispelled by recalling the separate roles played by the classical evolution operator $U(\sigma') = \exp(-\sigma'\, ad\, I')$, $ad\, I' \equiv [I',\ ]_{P.B.}$, and the dynamical invariance group. The former, applied to a set of initial conditions, generates a trajectory. The latter, applied to a trajectory, yields another trajectory that is a solution of the dynamics. Thus the dynamical invariance group acts as a permutation group on the set of initial conditions, and it is only rarely that a dynamical trajectory is the result of a one-parameter subgroup of such permutations. This is illustrated by the Kepler problem, which has 0(3; R) as a constant of the motion, yet admits elliptical trajectories.

2.6. Constants of the Motion

The phase space has twenty-eight dimensions, so there are twenty-seven constants of the motion.

Six of these are given by the vectors B^μ and C^μ of Eq. (14). These two four-vectors constitute only six independent quantities, since they are each perpendicular to p_μ.

A seventh is the invariant $p_\alpha L^\alpha$.

Ten more are provided by a canonical realization of ilg, the Lie algebra of ILG:

$$P_\mu = p_\mu \tag{17a}$$

$$H_{\mu\nu} = x_{[\mu}\, p_{\nu]} + L_{\mu\nu} \tag{17b}$$

The remaining ten constants of the motion are the quantities R_{ab}, the generators of right translation. Their Poisson brackets with the x^μ and p_μ vanish trivially, while their brackets with the L_{ab} vanish by the associative law of group multiplication.

In addition to these continuous symmetries, the dynamical generator is invariant under certain discrete operations. In preparation for this, consider a one-dimensional dynamical system with canonical coordinates x and p. Let Λ be the operation that reflects x. Then $\Lambda x = -x,\ \Lambda x^2 = x^2,\ \Lambda x^3 = -x^3, \ldots$; and $\Lambda p = -p,\ \Lambda p^2 = p^2,\ \Lambda p^3 = -p^3, \ldots$. These equations are consequences of the classical commutation relations $\Lambda x = -x\Lambda$ and $\Lambda p = -p\Lambda$.

Passing now to the bundle, we introduce five involutions Λ^a, with $\Lambda_a = g_{ab}\,\Lambda^b$. Λ^a is that operation on the bundle that is associated with reflection of the coordinate x^a. In particular, it reverses the ath axis of the fiducial frame. The classical commutation relations now become $\Lambda^a\, x^b = (-)^{g^{ab}}\, x^b\, \Lambda^a$ and $\Lambda^a\, p_b = (-)^{\delta^a_b}\, p_b\, \Lambda^a$. Acting on the orbital generators $J^{bc} \equiv X^{[b\, p^c]}$, these imply $\Lambda^a\, J^{bc} = (-)^{g^{ab}+g^{ac}}\, J^{bc}\, \Lambda^a$. By cogredience, $\Lambda^a\, L^{bc} = (-)^{g^{ab}+g^{ac}}\, L^{bc}\, \Lambda^a$. In particular, Λ^5 anti-commutes with the four $L^{5\mu}$, and is the only reflection to do so.

The five reflections Λ^a are somewhat redundant, for reflection of x^μ is equivalent to performing a π rotation in the 5μ plane, and then reversing the fifth axis. I.e., $\Lambda^\mu = \Lambda^5 \exp \pi\, ad(J^{5\mu} + L^{5\mu})$.

One then checks that the dynamical generator I' is invariant under the four reflections Λ^μ, while under Λ^5,

$$\left[L^{5\mu}\, p_\mu - \frac{L_{ab}\, L^{ab}}{2R} - \frac{R}{2}(m_0 c)^2\right] \to \left[-L^{5\mu}\, p_\mu - \frac{L_{ab}\, L^{ab}}{2R} - \frac{R}{2}\,(m_0 c)^2\right].$$

2.7. Interactions

The simplest interactions are those involving external fields, and of these the electromagnetic field is of greatest importance. Denoting the free Lagrangian by $\mathcal{L}_0$ the corresponding action is

$$S = \int d\sigma\, \mathcal{L}_0 + \frac{e}{c}\int d\sigma\, \dot{x}^\mu\, A_\mu \tag{18}$$

which is gauge invariant and homogeneous of degree one. As in Section 4, the canonical momenta derived from this action are not all independent, being subject to the constraint $L^\mu(p_\mu - \frac{e}{c}\, A_\mu) - \frac{L_{ab}\, L^{ab}}{2R} - \frac{R}{2}\,(m_0 c)^2 = 0$. Thus the inclusion of the electromagnetic field is accounted for by the transition

$$I(x, \rho, q, r) \to I\left(x, p - \frac{e}{c}\, A, q, r\right) \equiv I(e).$$

One notes from Eq. (18) that, on introduction of the electromagnetic field, the former total invariance of the theory under reparametrization is reduced to invariance under sense-preserving reparametrizations. Indeed, if $\sigma \to -\sigma$,

$$S \to -\int d\sigma\, \mathcal{L}_0 + \frac{e}{c}\int d\sigma\, \dot{x}^\mu\, A_\mu,$$

in which the sign of the charge has been effectively reversed. (This same effect is induced by changing to the negative branch of the square root in the free Lagrangian.)

In the Hamiltonian formulation, the reversal $\sigma \to -\sigma$ is equivalent to reversing the fundamental Poisson bracket relations:

$$[x^\mu, p_\nu]_{P.B.} = \delta^\mu_\nu \to [p_\nu, x^\mu]_{P.B.} = \delta^\mu_\nu, \quad \text{and}$$
$$[q^{ab}, r_{cd}]_{P.B.} = \delta^{ab}_{cd} \to [r_{cd}, q^{ab}]_{P.B.} = \delta^{ab}_{cd}.$$

This reversal is induced by the anti-canonical transformation

$$C_0 : \begin{array}{lll} x^\mu \to & x^\mu & \qquad q^{ab} \to \quad q^{ab} \\ p_\mu \to & -p_\mu & \qquad r_{ab} \to -r_{ab} \end{array} \tag{19}$$

Since the L^{ab} are linear in the r's, $C_0 L^{ab} = -L^{ab}$, which induces $C_0\, I(e) = I(-e)$.

Let now $C_1^\pm$ be any two canonical transformations preserving the forms of $I(\pm e)$, respectively. Then the composite anti-canonical transformation $C \equiv C_1^- \, C_0 \, C_1^+$ qualifies for the title, "charge conjugation," since C applied to a solution of the $I(e)$ dynamics yields a solution of the $I(-e)$ dynamics. Its characteristic feature is that a negative energy electron, traveling backward into the past, is converted into a positive energy positron, going forward into the future [It is an interesting question, to what extent the involutive automorphisms of the quantum theory are mirrored in the symmetries known to exist on the group. For the geometry of Lie groups and their coset spaces see (Lechnerowicz, 1958; Helgason, 1962)].

The presence of an electromagnetic field produces an interesting effect on the motion of a formerly free particle. It was seen in Section 5 that the projection of the trajectory onto the base space was trochoidal, being built up of oscillations and translations. The oscillations were governed by an isotropic frequency $\overline{p}$. Introducing an electromagnetic field destroys this isotropy, so that the oscillatory part of the motion becomes, for weak fields, a generally incommensurable Lissajous figure.

The study of the interacting dynamics of particles and the electromagnetic field may be based on the action

$$S = \sum_i \int d\sigma_i \; \mathcal{L}_0(i) + \sum_i \frac{e_i}{c} \int d\sigma_i \; \dot{x}^\mu(i)\, A_\mu - \frac{1}{4} \int d^4 x \; A_{[\mu,\nu]} \; A^{[\mu,\nu]} \tag{20}$$

This classical theory, which may perhaps be considered the antecedent of quantum electrodynamics, is already of enormous complexity, and will not be further treated here.

There is a peculiarity implicit in Eq. (20) that seems to be characteristic of relativistic systems. This peculiarity may be illustrated by the following two examples.

The first example consists of two free particles. The action is $S = \int d_1^\sigma \; \mathcal{L}_0(1) + \int d\sigma_2 \, \mathcal{L}_0(2)$. Computing the canonical momenta, one finds there are now two constraints: $I(1) = 0$ and $I(2) = 0$. Alternatively, $E_1 = 0$ and $E_2 = 0$, where E_1 and E_2 are the elementary symmetric polynomials in the I's. This feature extends to a many-particle system, there being as many constraints as particles.

Even on introducing particle-particle interactions, this feature seems to persist, as the next example shows. Consider two particles interacting with the electromagnetic field according to Eq. (20). It is known (Wheeler and Feynman, 1949) that, as far as a designated subset of particles is concerned, the electromagnetic field may plausibly be eliminated in favor of an induced action at a distance between these particles. This induced theory is given by the Fokker action

$$\begin{aligned} S = \int d\sigma_1\, \mathcal{L}_0(1) + \int d\sigma_2\, \mathcal{L}_0(2) + \frac{e_1\, e_2}{c^2} \\ \times \int d\sigma_1\, d\sigma_2\, \delta((x(1) - x(2))^2)\dot{x}^{\mu}(1)\, \dot{x}_{\mu}(2) \end{aligned} \tag{21}$$

Computing the canonical momenta leads to the two non-local constraints

$$\begin{aligned} I\left(x(1),\ p_\mu(1) - \frac{e_1\, e_2}{c^2}\int d\sigma_2\, \delta((x(1) - x(2))^2)\dot{x}_\mu(2),\ q(1),\ r(1)\right) = 0 \\ I\left(x(2),\ p_\mu(2) - \frac{e_1\, e_2}{c^2}\int d\sigma_1\, \delta((x(1) - x(2))^2)\dot{x}_\mu(1),\ q(2),\ r(2)\right) = 0 \end{aligned} \tag{22}$$

in which the velocities have not yet been eliminated. It is conjectured that the resulting motion may be referred to a single invariant parameter, and that evolution with respect to this parameter is governed by two invariant generators [The analogous problem of finding the single covariant generator for spinless action-at-a-distance theory has been solved by Kerner (1962)], of a form derived in a related context by Schwinger (1951).

If it is true that relativistic many-body systems will generally lead to multiple constraints, it would appear that the correct quantum treatment of interactions is far more complex than has been imagined. This may be inferred from the following partial summary of a procedure, due to Dirac 1964, for handling multiple constraints.

Suppose one were to start with a Lagrangian and attempt to turn the canonical crank. This is possible only if the equations defining the canonical momenta are solvable for the velocities. If this inversion is not possible, it must be that there are relations between the coordinates and momenta, called primary constraints. (E.g., in Section 4 of this paper there was one primary constraint, $I = 0$.) Let there be j primary constraints: $I_1 = 0, \ldots, I_j = 0$. Then, by Hamilton's principle, the motion possesses the generator $G = H + \lambda_1\, I_1 + \cdots + \lambda_j\, I_j$ involving a non-unique Hamiltonian and j Lagrange multipliers. (In our example H vanished, since the Lagrangian was homogeneous of degree one, so $G = \lambda I$.) One must require that the vanishing of the constraints be preserved by the dynamics. This is a compatibility condition, and in general requires the vanishing of further expressions: $I_{j+1} = 0, \ldots, I_k = 0$; which, in turn, must be compatible with the dynamics. In this way, the scheme is prolonged until one reaches a full constraints ideal: $I_1 = 0, \ldots, I_\ell = 0$. By construction, this ideal exhausts the requirements imposed by the dynamics.

On passing to the quantization of the system, these constraints become subsidiary conditions to be satisfied by the state ray: $\langle\psi|I_1 = 0, \ldots, \langle\psi|I_\ell = 0$. Consider two of these equations, say $\langle\psi|I_1 = 0$ and $\langle\psi|I_2 = 0$. Then certainly $\langle\psi|I_1\, I_2 = 0$ and $\langle\psi|I_2\, I_1 = 0$; hence $\langle\psi|[I_1,\ I_2] = 0$. In general, the commutator of two constraints does not lie in the constraint ideal, so $\langle\psi|[I_1, I_2] = 0$ is a new and independent condition. But, by construction, the ideal was the full expression of the restrictions imposed by the dynamics. Thus one must group together those constraints whose commutators with the ideal lead back into the ideal, and those which do not.

These two pieces constitute, respectively, the first class and second class parts of the ideal. In order to avoid the above contradictions, the second class part of the ideal must effectively be eliminated. This is done by performing a contact transformation, where possible, or through the use of Dirac brackets.

Let us apply this formalism to the case of n free particles. We assume, as in the theory of this paper, that the Hamiltonian vanishes, and that there are n constraints, one for each particle. The wave function for this n body system then satisfies the Schrödinger equations $\langle\psi|I_1 = 0, \ldots, \langle\psi|I_n = 0$. Since the particles are uncoupled, the constraints have no variables in common. Hence the full constraint ideal is first class. (It is interesting to note that the constraints permute among themselves under the action of the symmetric group. Then if $\langle\psi|$ is taken to be purely symmetric or antisymmetric, or a linear combination of these symmetry types, $\langle\psi|$ need only be annihilated by any one constraint in order to be annihilated by them all.)

When one passes to interacting particles, the constraints will have variables in common. One may then anticipate the possible occurrence of second class constraints, necessitating the special procedures of Dirac's formalism. In view of the difficulties that have inhibited the successful treatment of interactions in quantum field theory, it may well be that this complication is a blessing in disguise.

Lastly, we consider point interactions between two particles. We may write the five tensors $(1, \dot{x}^\mu, W_{5\mu\, ab}\, \dot{q}^{ab}, W_{\mu\nu ab}\, \dot{q}^{ab}, \epsilon_{5\mu\nu\sigma\tau}\, W^{\sigma\tau}_{ab}\, \dot{q}^{ab})$, one set for each of the two particles, and construct invariants whose forms begin to mimic the SVTAP interactions of quantum field theory. Then, presumably, these invariants would appear in an interaction Lagrangian preceded by a function which pulses on contact of the particles:

$$\mathcal{L}_{INT} = \int\int d\sigma_1 d\sigma_2\, \delta\,(x_1(\sigma_1), x_2(\sigma_2)) \times \text{an invariant.}$$

However, it is not possible to do this in a Lorentz-invariant way. For contact is associated with the apex of the light cone, and this can only be singled out by a four-dimensional δ function: $\delta^{(4)}(x_1, x_2) = \delta(x_1^0 - x_2^0)\,\delta(x_1^1 - x_2^1)\,\delta(x_1^2 - x_2^2)\,\delta(x_1^3 - x_2^3)$. The appearance of such a function is meaningless in the resulting Euler-Lagrange equations, which are ordinary differential equations in the independent variables σ_1 and σ_2. To secure a meaningful expression (even in the sense of generalized functions), requires a one-dimensional δ function, whose argument, to secure Lorentz invariance, is invariant. But the only invariant that may be formed from x_1 and x_2 is $(x_1 - x_2)^2$, which singles out the entire light cone, not just its apex. Thus, within the context of a classical Lorentz-invariant theory, one concludes that it is impossible to formulate point interactions. Physically, this is not surprising: one should expect particles to interact not when their base points coincide (a contingency of measure zero), but when their frames bang into each other.

3. QUANTUM THEORY

3.1. Quantization

Quantization is achieved by promoting the dynamical variables to operators (e.g., $x^\mu \to \hat{x}^\mu$), acting on an initially unspecified Hilbert space. The operators are defined by their commutation relations, obtained from the fundamental Poisson bracket relations by the replacement

$$\begin{aligned} [x^\mu, p_\nu]_{P.B.} = \delta^\mu_\nu &\longrightarrow [\hat{x}^\mu, \hat{p}_\nu] = i\hbar\,\delta^\mu_\nu \\ [q^{ab}, r_{cd}]_{P.B.} = \delta^{ab}_{cd} &\longrightarrow [\hat{q}^{ab}, \hat{r}_{cd}] = i\hbar\,\delta^{ab}_{cd} \end{aligned} \tag{23}$$

The form I now becomes a quantum operator via $I(x, p, q, r) \to \hat{I} \equiv I(\hat{x}, \hat{p}, \hat{q}, \hat{r})$. One then introduces an abstract state $\langle\psi|$. The classical constraint $I = 0$ passes over to a condition on this state, by writing the left Schrödinger equation $\langle\psi|\hat{I} = 0$:

$$\langle\psi|\left(\hat{L}^\mu\,\hat{p}_\mu - \frac{\hat{L}_{ab}\,\hat{L}^{ab}}{2R} - \frac{R}{2}(m_0c)^2\right) = 0. \tag{24}$$

3.2. Reduction of Representations

In determining the states, one may proceed by the method of Schrödinger. Realize the commutation relations of Eq. (23) by $\hat{x} \to x$, $\hat{p} \to \frac{\hbar}{i}\frac{\partial}{\partial x}$, $\hat{q} \to q$, $\hat{r} \to \frac{\hbar}{i}\frac{\partial}{\partial q}$. Then expand the fiber dependence of the left state $\langle\psi|$ in terms of those harmonics on the frame manifold that are the substrata of the finite dimensional, non-unitary representations of 0(3, 2; R). The "boundary condition" that selects these harmonics is, apparently, the demand that the irreducible representations of 0(3, 2; R) be real, rather than Hermitian.

Proceeding in this way, or abstractly from the commutation relations, the Schrödinger equation breaks up into the direct sum of irreducible equations. These are the Dirac, Klein-Gordon, Maxwell-Proca, and higher spin equations (Corson, 1953; Schweber, 1961; Rose, 1961). For this reduction, we refer to the paper of Bhabha (Klein, 1936; Bhabha, 1945), who discusses the structurally similar case of the representations of 0(4, 1; R).

The left states $\langle\psi|$ are labelled by the eigenvalues of a maximal set of commuting constants of the motion. There are thirteen such operators.

One is given by $\hat{L}^{\alpha}\,\hat{p}_{\alpha}$.

Six are derived (Wigner, 1939; Bargmann and Wigner, 1948) from ilg: $\hat{p}_{\mu}$, $\widehat{W}^2/\hat{p}^2$, and $\widehat{W}_0$; where $\widehat{W}_{\mu} \equiv \frac{1}{2}\epsilon_{5\mu\nu\sigma\tau}\,\widehat{H}^{\nu\sigma}\,\hat{p}^{\tau}$.

The remaining six are obtained from the right translations:

$$\hat{\underline{S}}^2 \equiv \frac{1}{2}\,\delta^{50ab}_{50cd}\,\widehat{R}_{ab}\,\widehat{R}^{cd};\; \hat{S}_3 \equiv R_{12};\; \hat{\underline{S}}^2 - \hat{\underline{K}}^2 \equiv \frac{1}{2}\,\delta^{5ab}_{5cd}\quad \widehat{R}_{ab}\,\widehat{R}^{cd};\, \hat{\underline{S}}\cdot\hat{\underline{K}}$$

$$\equiv \frac{1}{2}\,\epsilon_{5\mathrm{abcd}}\quad \widehat{R}^{ab}\,\widehat{R}^{cd};$$

and the Casimir operators $\widehat{C}_1 \equiv \widehat{R}_{ab}\,\widehat{R}^{ab}$, and $\widehat{C}_2 \equiv \delta^{\mathrm{abcd}}_{\mathrm{efgh}}\,\widehat{R}_{ab}\,\widehat{R}_{cd}\,\widehat{R}^{ef}\,\widehat{R}^{gh}$.

If the quantum number derived from $\hat{L}^{\alpha}\,\hat{p}_{\alpha}$ vanishes, there is a certain degeneracy. It was seen in Section 6 that the form I was invariant under Λ^5 in precisely this case. One may then classify the states in terms of their behavior under reflection of the fifth axis. The states that are odd under this involution then effectively satisfy the equation of Weyl (Corson, 1953; Schweber, 1961; Rose, 1961).

3.3. Expectation Values

It has seemed that calculations involving average values must invoke quantities pertaining to the external observer. We assume that, for a non-rotating observer, these quantities are the four-vector $\mathcal{P}_{\mu}$, describing the progress of the external macroscopic observer through space-time. Thus, for an observer of mass M at rest, $\mathcal{P}_{\mu} = (Mc, 0, 0, 0)$.

It is further assumed that the participation of the external observer is effected by the introduction of a metric operator $\widehat{N}$ given by

$$\widehat{N} = \frac{\hat{L}^{\alpha}\,\mathcal{P}_{\alpha}}{\hbar\sqrt{\mathcal{P}^{\mu}\,\mathcal{P}_{\mu}}}. \tag{25}$$

Also associated with the external observer is the pseudo-scalar 3-volume $dV = \epsilon_{5\mu\nu\sigma\tau}\,\frac{\mathcal{P}^{\mu}}{\sqrt{\mathcal{P}^{\alpha}\,\mathcal{P}_{\alpha}}}\,dx^{\nu}\,dx^{\sigma}\,dx^{\tau}$. The expection value of an operator $\widehat{O}$, in the state

$\langle\psi|$, is given by the formula

$$\langle\widehat{O}\rangle_\psi = \int_V dV \langle\psi|\widehat{O}\,\widehat{N}|\psi\rangle \tag{26}$$

where the bracket denotes an averaging over the internal manifold.

As an example, consider an observer at rest measuring the electric current four-vector $\hat{j}^\mu$ (the promotion of the classical $j^\mu = \frac{e}{c}\frac{dx^\mu}{d\sigma} = \frac{e}{c}\,L^\mu$), in a state of spin 1/2:

$$\begin{aligned}\langle\hat{j}^\mu\rangle_{\frac{1}{2}} &= \int_V dV \left\langle \frac{1}{2} \left| \frac{e}{c}\,\hat{L}^\mu\,\frac{\hat{L}^\alpha\,\mathcal{P}_\alpha}{\hbar\sqrt{\mathcal{P}^\beta\,\mathcal{P}_\beta}} \right| \frac{1}{2} \right\rangle \\ &= \int_{t=\text{const}} d^3x\,\frac{e}{\hbar\,c}\,\psi\,\gamma^\mu\,\gamma^0\,\psi^\dagger.\end{aligned}$$

3.4. Mass Formula

The mass of a state $\langle\psi|$ is the expectation value of the operator $\widehat{m} = \pm\frac{1}{c}(\hat{p}^\mu\,\hat{p}_\mu)^{1/2}$. This invariant is most easily evaluated in the "rest" state of the particle, defined by the vanishing of its three spatial momenta. In this state, $\widehat{m} = \pm\hat{p}_0$. Solving Eq. (24) for $\hat{p}_0$ yields

$$\langle\widehat{m}\rangle_\psi = \pm\frac{1}{c}\int_V dV \left\langle \psi \left| \frac{1}{\hat{L}_0}\left(\frac{R}{2}(m_0c)^2 + \frac{\hat{L}_{ab}\,\hat{L}^{ab}}{2R}\right)\widehat{N} \right| \psi \right\rangle. \tag{27}$$

For an observer at rest,

$$\langle\widehat{m}\rangle_\psi = \pm\frac{1}{\hbar\,c}\int_{t=\text{const}} d^3x \left\langle \psi \left| \frac{R}{2}(m_0c)^2 + \frac{\hat{L}_{ab}\,\hat{L}^{ab}}{2R} \right| \psi \right\rangle. \tag{28}$$

leading to a rotational spectrum which may bear some relation to the "resonances" currently being discovered.

Such a formula cannot lead to the baffling mass spectrum presented by the elementary particles. It is true that the above mass formula would undergo modification on introducing interactions. It is felt, however, that the gross features of the elementary particle spectrum are attributable to an internal binding far stronger than the splitting provided by external interactions. It is useful to recall that in hydrogen, the energy is almost wholly dependent on the radial quantum number, associated with a coordinate untouched by $0(3; R)$, the *a priori* system group of the problem. In the context of the present theory, there are sixteen such variables: x^5; and the fifteen dimensions of $GL(5; R)/0(3, 2; R)$, corresponding to those elastic deformations in five-space that have not been considered.

4. DISCUSSION

The theory presented above was based on a simple, phenomenological Lagrangian, whose chief structural feature was the coupling of the 5μ components of the Darboux-Cartan matrix to the space-time velocities. This was made possible by the fact that HLG is a reductive subgroup of $0(3, 2; R)$. We have stressed the interpretation of the fiber as a manifold of orientation-deformations. This leads one to view the theory as describing the propagation of localized strains. I.e., in some sense, one may be studying the particle-like excitations of a still deeper field, which, for the sake of historical continuity, we shall call the ether. It is worth noting that the matrices of $0(3, 2; R)$, or rather its proper component, are unimodular; precisely what would be expected in a first, approximate description of an extremely stiff "luminiferous ether."

The coupling of a fifth axis quantity to space-time velocities is reminiscent of the treatment of the vector potential in the Kaluza-Klein (For a critical discussion and references, see Pauli, 1958) five dimensional formulation of general relativity. There also, as here, the fifth coordinate is ignorable in the Lagrangian, but plays an essential kinematic role. The esthetic appeal of this formulation, and others, would tend to suggest that Einstein's great discovery is not a closed and finished structure; but that, as with all things of depth, it is susceptible of fruitful generalization. It would be in the spirit of the present theory to seek an extension of general relativity from the base space to the bundle. In this connection, it is notable that the Lie manifolds satisfy Einstein's source-free equation.

In conclusion, the idea that motivated the present work may be formalized as the "Family Principle" concerning linear partial differential equations with constant coefficients. On the basis of the preceding, one may believe that linear partial differential equations with constant coefficients properly come in families, rather than singly, a family constituting of well-defined series of irreducible representations of a common invariant group; and that this entire family may be regarded as the Schrödinger field associated with one underlying system of ordinary differential equations. This might apply to equations with variable coefficients, provided one were able to attribute the variability to curvature of the base space.

An example is afforded by the Cauchy-Riemann equations, relating the real and imaginary parts of an analytic function. The base space is the Minkowski plane (the imaginary axis corresponding to time), the group and fiber $0(1, 2; R)$, the dynamical invariance group the inhomogeneous Lorentz group of the plane. The Lagrangian is that of Eq. (8), with $a, b = 3, 0, 1$ and $\mu, \nu = 0, 1$. The Schrödinger equation is then $\langle\psi|(\hat{L}^{3\mu}\,\hat{p}_\mu - \frac{\hat{L}_{ab}\hat{L}^{ab}}{2R} - \frac{R}{2}\,(m_0c)^2) = 0$. We examine the spin $1/2$ case. Choosing a real representation of the spin $1/2$ matrices, with L^{31} diagonal (corresponding to choice of x^1 as the real axis), and letting $R = \frac{1}{2}\,\frac{\hbar}{m_0c}$, yields in conventional notation,

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{\partial}{\partial x} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{\partial}{\partial y}\right) \begin{bmatrix} \mu \\ \nu \end{bmatrix} = 0,$$

the Cauchy-Riemann equations.

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